CE 228N: Introduction to the Theory of Plasticity: Homework III

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1. The invariants of a second order tensor can be considered as scalar functions of the tensor. In this context find the partial derivatives
   (i) $\dfrac{\partial I_1}{\partial \boldsymbol{\sigma}}$ (ii) $\dfrac{\partial J_3}{\partial \mathbf{S}}$ (iii) $\dfrac{\partial J_3}{\partial \boldsymbol{\sigma}}$
   Also find the component forms of these expressions, e.g. $\dfrac{\partial I_1}{\partial \sigma_{ij}}$.

2. Consider the flow rule associated with the J_2 yield criterion, namely $\dot{\varepsilon}^p_{ij} = \mathring{\lambda} S_{ij}$

   Explain why this can also be expressed as $\dot{\varepsilon}^p_{(i)} = \mathring{\lambda} S_{(i)}$ where the parenthetical indices refer to principal components in a shared eigenbasis.

   Is it meaningful to further write $\dot{\varepsilon}'^p_{(i)} = \mathring{\lambda} S'_{(i)}$, where the primes refer to principal components in the transformed $\sigma'_1, \sigma'_2, \sigma'_3$ basis in the Haigh-Westergaard space? Explain your answer.

3. Show that the plastic dissipation associated with the Tresca yield criterion is

$$D_p(\dot{\varepsilon}^p) = \mathring{\lambda} k = k\left(|\dot{\varepsilon}^p_1| + |\dot{\varepsilon}^p_2| + |\dot{\varepsilon}^p_3|\right)$$

   Note that for Tresca yield with hardening, it is more appropriate to define the Tresca equivalent plastic strain as

$$\bar{\varepsilon} = \frac{1}{2}\int \left(|\dot{\varepsilon}^p_1| + |\dot{\varepsilon}^p_2| + |\dot{\varepsilon}^p_3|\right) dt$$

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and then use this $\bar{\varepsilon}$ as a hardening variable to specify the evolution of k, i.e. $k = k(\bar{\varepsilon})$ rather than the von Mises $\bar{\varepsilon}$.

4. Show that the norm induced by the tensor inner product $||\mathbf{S}|| = \sqrt{\mathbf{S} : \mathbf{S}}$ indeed satisfies all the requirements of a norm.

5. Consider a von Mises rate-independent plastic material with isotropic hardening; this material has a yield criterion

$$f(\boldsymbol{\sigma}, \overline{\boldsymbol{\varepsilon}}) := \sqrt{J_2} - k(\overline{\boldsymbol{\varepsilon}}) = 0$$

Assuming an associated flow rule $\dot{\varepsilon}^p_{ij} = \overset{\circ}{\lambda} S_{ij}$, show that

$$\dot{\varepsilon}^p_{ij} = \frac{\sqrt{3} S_{ij} \left\langle S_{pq} \dot{S_{pq}} \right\rangle}{4k^2\, k'(\overline{\boldsymbol{\varepsilon}})}$$

Explain your reasoning.

6. For the same material as in the previous question, show that

$$\dot{\varepsilon}^p_{ij} = \frac{S_{ij} \left\langle S_{pq} \dot{\varepsilon}_{pq} \right\rangle}{2k^2 \left(1 + \dfrac{k'(\overline{\boldsymbol{\varepsilon}})}{\sqrt{3}\mu} \right)}$$

where μ is the shear modulus.

7. Recall the definition of the elastic-plastic tangent modulus

$$\mathbb{C}^{ep} : \dot{\boldsymbol{\varepsilon}} = \mathbb{C} : \dot{\boldsymbol{\varepsilon}} - \overset{\circ}{\lambda}\, \mathbb{C} : \mathbf{r}$$

Recreate the proof of the statement that

$$\mathbb{C}^{ep} = \mathbb{C} - \frac{\mathbb{C} : \mathbf{r} \otimes \mathbb{C} : \partial_\sigma f}{\partial_\sigma f : \mathbb{C} : \mathbf{r} + \partial_q f \cdot \mathbf{h}}$$

is a solution to this equation. Assume Simo's expression for $\overset{\circ}{\lambda}$ given in class.

The critical step is to establish the following identity:

$$\mathbb{C} : \mathbf{r} \otimes \mathbb{C} : \partial_\sigma f : \dot{\boldsymbol{\varepsilon}} = (\partial_\sigma f : \mathbb{C} : \dot{\boldsymbol{\varepsilon}})\, \mathbb{C} : \mathbf{r}$$

8. Let $\mathbf{n} = \dfrac{\mathbf{S}}{||\mathbf{S}||}$. Evaluate the expression

$$\frac{\mathbf{n} \colon \mathbb{C} \colon \dot{\boldsymbol{\varepsilon}}}{\mathbf{n} \colon \mathbb{C} \colon \mathbf{n}}$$

for an isotropic material as done in class, justifying each step.

9. Using Simo's expression for $\mathring{\lambda}$, find an expression for the plastic strain rate tensor for an isotropically hardening von Mises material with linear hardening modulus H. Show that this result is consistent with the expression in Q6.

10. Consider a rate independent material which shows pure kinematic hardening, i.e.

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}) := \sqrt{\frac{1}{2}(\mathbf{S} - \boldsymbol{\alpha}) : (\mathbf{S} - \boldsymbol{\alpha})} - k = 0$$

Assuming Melan-Prager evolution of the back stress tensor $\boldsymbol{\alpha}$, show that

$$\dot{\boldsymbol{\alpha}} = \frac{(\mathbf{S} - \boldsymbol{\alpha}) : \dot{\mathbf{S}}}{2k^2} (\mathbf{S} - \boldsymbol{\alpha})$$